5. Roots and weights
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The discussion here largely follows chapters V–VII of [1].

5.1 Representations I

5.1.1 Matrix representations and weight vectors

In module 4, we discussed how a general Lie algebra L expressed in the Cartan-Weyl basis is a complex of interwoven su(2)s, formed from raising and lowering operators (root vectors $\{e_\alpha, e_{-\alpha}\}$) and a basis $\{h_i\} \in H$ of mutually commuting Cartan subalgebra elements, $[h_i, h_j] = 0$. The number of independent basis elements for H is the rank r, and the group dimension $d = r + 2 \times$ (# of positive root vectors).

For each pair of raising and lowering operators, we can construct a unique element $h_\alpha \in H$ that serves as the "S_z" for that particular su(2): [c.f. Eq. (4.4.11)]

$$[h_\alpha, e_\alpha] = \alpha(h_\alpha)\, e_\alpha = \langle \alpha, \alpha \rangle\, e_\alpha, \tag{5.1.1a}$$

$$[h_\alpha, e_{-\alpha}] = -\,\alpha(h_\alpha)\, e_{-\alpha} = -\langle \alpha, \alpha \rangle\, e_{-\alpha}, \tag{5.1.1b}$$

$$[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})\, h_\alpha. \tag{5.1.1c}$$

Here $(e_\alpha, e_{-\alpha})$ is the Killing form for this pair of raising and lowering operators; the last equation defines the H element h_α. The root $\alpha = \alpha(h)$ belongs to the **space of weights** H^*, and is a functional on a generic Cartan subalgebra element $h \in H$. When we choose $h = h_\alpha$, the resulting eigenvalue is the weight space inner product [Eq. (4.3.6)]

$$\alpha(h_\alpha) = (h_\alpha, h_\alpha) = \langle \alpha, \alpha \rangle, \qquad H_0^* \times H_0^* \to \mathbb{R}.$$

Recall that while we can construct an h_α for every root vector pair $\{e_\alpha, e_{-\alpha}\}$ in L, this set is overcomplete [unless $r = 1$, su(2)]. In the Cartan subalgebra, it is important not to confuse the overcomplete $\{h_\alpha\}$ with the basis $\{h_i\}$; a prescription for constructing the latter will be described later.

Now, consider an N-dimensional irreducible unitary matrix representation of L, so that

$$a \in L \to \hat{A}, \quad \text{an } N \times N \text{ matrix.}$$

The Cartan subalgebra elements map to

$$[h, h'] \to [\hat{H}, \hat{H}'] = 0. \tag{5.1.2}$$

In a unitary representation, we can without loss of generality (WLoG) assume that each basis element $h_i \in H$ $(1 \leq i \leq r)$ corresponds to a Hermitian matrix $\hat{H}_i = \hat{H}_i^\dagger$. Then there is a basis for vectors in the representation that diagonalizes the mutually commuting $\{\hat{H}_i\}$. An element of this basis is a **weight vector** ϕ^a, where $0 \leq a \leq N-1$ (N is the dimension of the representation). In the eigenbasis of $\{H_i\}$, we can take (e.g.)

$$\phi^0 \to \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \phi^1 \to \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots \quad \phi^{N-1} \to \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \tag{5.1.3}$$

A weight vector satisfies

$$\hat{H}_i \, \phi^a = \lambda_i^a \, \phi^a, \quad 1 \leq i \leq r, \tag{5.1.4}$$

where the eigenvalue $\lambda_i^a \in \mathbb{R}$. Thus each weight vector is specified by the set of eigenvalues obtained by acting with any of the $\{\hat{H}_i\}$. This is similar to quantum mechanics, where one seeks to identify a physical state by labeling it via a complete set of commuting observables [2], e.g. mutually diagonalizable Hermitian matrices. Even in a given irreduible representation, however, a weight vector labeled by $\{\lambda_i^a\}$ is not necessarily unique. Later we will show that there can be degeneracies; this is different from the representation theory of su(2).

Instead of employing the explicit $\{\lambda_i^a\}$ obtained in some particular basis, we recast Eq. (5.1.4) for a generic Cartan subalgebra element $h \to \hat{H}$:

$$\hat{H} \, \phi^a = M^a(h) \, \phi^a, \tag{5.1.5}$$

where the functional $M^a(h)$ is called the **weight**, and resides in the same space H^* as the roots.

• <u>su(3) example</u>: A basis for H in the defining representation is given by [Eqs. (3.1.4) and (3.1.1)]

$$\hat{T}_z = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{Y} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{bmatrix}. \tag{5.1.6}$$

In this basis, we denote the weight vectors as

$$\phi^A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \phi^B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \phi^C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{5.1.7}$$

Then

$$\begin{aligned} M^A(at_z + by) &= \frac{1}{2}a + \frac{1}{3}b &&= \frac{2}{3}\alpha_{t_+} + \frac{1}{3}\alpha_{u_+} \\ M^B(at_z + by) &= -\frac{1}{2}a + \frac{1}{3}b &&= -\frac{1}{3}\alpha_{t_+} + \frac{1}{3}\alpha_{u_+} \\ M^C(at_z + by) &= -\frac{2}{3}b &&= -\frac{1}{3}\alpha_{t_+} - \frac{2}{3}\alpha_{u_+}. \end{aligned} \tag{5.1.8}$$

On the right-hand-sides, we have re-expressed the weights in terms of two of the positive roots using Eq. (3.2.9).

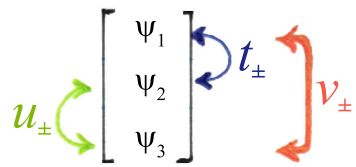

Figure 5.1: Action of root vector operators on a weight vector in the defining representation of su(3).

5.1.2 Raising and lowering; chains of weights

Assume that ϕ^a is a weight vector with weight M^a, Eq. (5.1.5). Then

$$\hat{H}\,\hat{E}_{\pm\alpha}\,\phi^a = \left[\hat{E}_{\pm\alpha}\,\hat{H} \pm \alpha(h)\,\hat{E}_{\pm\alpha}\right]\phi^a = \left[M^a(h) \pm \alpha(h)\right]\hat{E}_{\pm\alpha}\,\phi^a, \tag{5.1.9}$$

so that $\hat{E}_{\pm\alpha}\,\phi^a$, if non-zero, is a weight vector with weight $(M^a \pm \alpha)$.
This suggests the following picture: all weights belong to **weight strings** or chains, consisting of 1 or more weights separated by root intervals. These chains can be pictured as parallel 1D sublattices embedded in an r-dimensional Euclidean space. Acting with different pairs of root vectors, we translate through an r-dimensional **weight lattice**. Since the number of positive root vectors is greater than r for any $r > 1$, different sets of parallel sublattices generated by different roots cannot all be mutually orthogonal.

We consider two examples from su(3).

• Defining ("3") representation: The weights are given by Eq. (5.1.8), associated to the weight vectors in Eq. (5.1.7). The root vector operators have the matrix representations [Eqs. (3.1.4) and (3.1.1)]

$$\{\hat{T}_+, \hat{T}_-\} = \left\{\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}\right\}, \quad \{\hat{U}_+, \hat{U}_-\} = \left\{\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}\right\}, \quad \{\hat{V}_+, \hat{V}_-\} = \left\{\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}\right\}. \tag{5.1.10}$$

The action of these on a vector expressed in the weight vector basis is shown in Fig. 5.1. Consistent with this picture, Eq. (5.1.8) implies that

$$M^A = M^B + \alpha_{t_+}, \quad M^B = M^C + \alpha_{u_+}. \tag{5.1.11}$$

Fig. 5.1 also suggests that

$$M^A = M^C + \alpha_{v_+}. \tag{5.1.12}$$

Eqs. (5.1.11) and (5.1.12) then imply that

$$\alpha_{v^+} = \alpha_{t^+} + \alpha_{u_+}. \tag{5.1.13}$$

This is indeed correct, see Eq. (3.2.9). The defining representation can therefore be depicted as the triangle shown in Fig. 5.2.

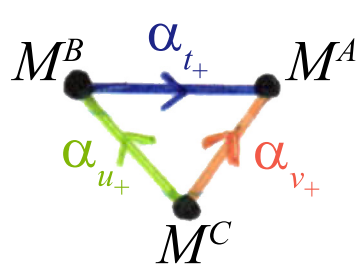

Figure 5.2: The defining representation of su(3).

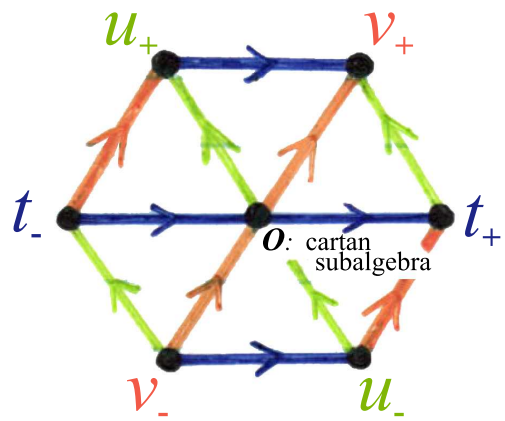

Figure 5.3: The adjoint representation of su(3).

• Adjoint representation: This is the representation in which the roots $\{t_\pm, u_\pm, v_\pm\}$ themselves are weights, along with a 2D basis for the Cartan subalgebra. Recall the commutation relations from module 3 between root vectors of different su(2)s, Eqs. (3.1.6) and (3.1.7):

$$\begin{aligned} ad_{t_+}(u_+) &= [t_+, u_+] &&= v_+, \\ ad_{t_+}(v_-) &= [t_+, v_-] &&= -u_-, \\ ad_{u_+}(v_-) &= [u_+, v_-] &&= t_-, \\ ad_{v_+}(t_-) &= [v_+, t_-] &&= -u_+, \\ ad_{v_+}(u_-) &= [v_+, u_-] &&= t_+. \end{aligned} \tag{5.1.14}$$

Previously we found that the root scalar product applied to su(3) implied the root geometry shown in Fig. 4.1 (where $\alpha_{\pm 1,2,3} \equiv \{\alpha_{t_\pm}, \alpha_{u_\pm}, \alpha_{v_\pm}\}$. Using the same color conventions as we used for $t_\pm$ (blue), $u_\pm$ (green), and $v_\pm$ (orange) "moves" in the defining representation [Fig. 5.2], Fig. 4.1 and Eq. (5.1.14) imply that the adjoint representation can be depicted as in Fig. 5.3. Note the following two points:

1. The coefficients (± 1) in Eq. (5.1.14) do not enter this geometric description of the representation. Only the "fusion rule" is important, i.e. whether the Lie bracket vanishes or not, and if not which root vector appears on the right-hand-side.

2. Given the root geometry from Fig. 4.1, Eq. (5.1.14) and Fig. 5.3 suggest that if $\{\alpha, \beta, \alpha+\beta\}$ are all roots, then $[e_\alpha, e_\beta] = N_{\alpha,\beta}\, e_{\alpha+\beta}$ *and* $N_{\alpha,\beta} \neq 0$. In other words, if $\alpha+\beta$ is a root, then we can access it by acting with ad_{e_α} on β or with ad_{e_β} on α. We might say that every possible move is allowed in the root system. You will prove this in general in a homework problem.

5.1.3 The Master Weight Depth Formula (MWDF)

Consider a highest weight vector ϕ^0 with weight M^*, in a weight string generated by acting with some particular root vectors $e_{\pm\alpha}$. Assume that the weight string terminates after lowering q times with $\hat{E}_{-\alpha}$, i.e. the weights are $\{M^*, M^*-\alpha, M^*-2\alpha, \ldots, M^*-q\alpha\}$. We will show that the depth q can be related to the inner product of the weight M^* with the root α.

Then

$$\hat{H}\,\phi^0 = M^*(h)\,\phi^0, \quad h \in H, \tag{5.1.15a}$$

$$\hat{E}_\alpha\,\phi^0 = 0, \tag{5.1.15b}$$

$$\hat{H}\,\left(\hat{E}_{-\alpha}\right)^j\,\phi^0 = [M^*(h) - j\,\alpha(h)]\left(\hat{E}_{-\alpha}\right)^j\,\phi^0, \qquad 0 \le j \le q, \tag{5.1.15c}$$

$$\left(\hat{E}_{-\alpha}\right)^{q+1}\,\phi^0 = 0. \tag{5.1.15d}$$

Assume for simplicity that a normalization for the root vector operators is chosen such that

$$(e_\alpha, e_{-\alpha}) = 1 \quad \Rightarrow \quad [e_\alpha, e_{-\alpha}] = h_\alpha.$$

This choice is always possible, and is obviously independent of the root normalization. Denote

$$\phi^j \equiv \left(\hat{E}_{-\alpha}\right)^j \phi^0, \qquad \hat{E}_\alpha \phi^j \equiv r_j \phi^{j-1}. \tag{5.1.16}$$

Then

$$\begin{aligned}\hat{E}_\alpha \phi^k &= r_k \phi^{k-1} = \hat{E}_\alpha \hat{E}_{-\alpha} \phi^{k-1} = \left(\hat{E}_{-\alpha} \hat{E}_\alpha + \hat{H}_\alpha\right) \phi^{k-1} = \{r_{k-1} + [M^*(h_\alpha) - \alpha(h_\alpha)(k-1)]\} \phi^{k-1} \\ &= [r_{k-1} + \langle M^*, \alpha\rangle - (k-1)\langle\alpha,\alpha\rangle] \phi^{k-1}, \end{aligned} \tag{5.1.17}$$

$$\begin{aligned}\hat{E}_{-\alpha} \phi^k &= \phi^{k+1} = \frac{1}{r_{k+1}} \hat{E}_{-\alpha} \hat{E}_\alpha \phi^{k+1} = \frac{1}{r_{k+1}} \left(\hat{E}_\alpha \hat{E}_{-\alpha} - \hat{H}_\alpha\right) \phi^{k+1} = \frac{1}{r_{k+1}} \{r_{k+2} - [M^*(h_\alpha) - \alpha(h_\alpha)(k+1)]\} \phi^{k+1} \\ &= \frac{1}{r_{k+1}} [r_{k+2} - \langle M^*, \alpha\rangle + (k+1)\langle\alpha,\alpha\rangle] \phi^{k+1}, \end{aligned} \tag{5.1.18}$$

so that

$$\begin{aligned} r_k &= r_{k-1} + \langle M^*, \alpha\rangle - (k-1)\langle\alpha,\alpha\rangle, \\ &= r_{k+1} - \langle M^*, \alpha\rangle + k\langle\alpha,\alpha\rangle. \end{aligned} \tag{5.1.19}$$

Since $r_0 = 0$, we obtain

$$r_k = k\,\langle M^*, \alpha\rangle - \frac{k(k-1)}{2}\langle\alpha,\alpha\rangle. \tag{5.1.20}$$

On the other hand,

$$\hat{E}_\alpha \phi^{q+1} = r_{q+1} \phi^q = 0$$

implies that $r_{q+1} = 0$, so that

$$r_q = -\langle M^*, \alpha\rangle + q\langle\alpha,\alpha\rangle. \tag{5.1.21}$$

Together Eqs. (5.1.20) and (5.1.21) lead to

$$q = \frac{2\langle M^*, \alpha\rangle}{\langle\alpha,\alpha\rangle}. \tag{5.1.22}$$

A more useful formula does not require that we start with the highest weight. Assume that weight M is somewhere in a string generated by $\hat{E}_\alpha$ (raising) and $\hat{E}_{-\alpha}$ (lowering) of length $m+p+1$, such that there are p weights (m weights) above (below) M. Then $q = m + p$, so that

- For weight M in the string of length $p + 1 + m$ $(m, p \geq 0)$,
  $\{M + p\alpha, M + (p-1)\alpha, \ldots, M + \alpha, M, M - \alpha, \ldots, M - (m-1)\alpha, M - m\alpha\}$,

$$m - p = \langle M, \alpha^\vee\rangle, \qquad \underline{\text{Master Weight Depth Formula}}\ \text{(MWDF)},$$

$$\alpha^\vee \equiv \frac{2\alpha}{\langle\alpha,\alpha\rangle}, \qquad \textbf{coroot}\ \text{associated to root}\ \alpha. \tag{5.1.23}$$

Eq. (5.1.23) is one of the most important formulae we will derive in this course. We have christened it the "Master Weight Depth Formula" (MWDF); apparently it has no proper appellation.[1] Since the product of any weight M and any **coroot** $\alpha^\vee$ is always an integer, one says that the coroots are dual to the weights.

• **Exercise**: Verify that Eq. (5.1.23) holds for $M = M^A$ in the defining representation of su(3), for two cases:

1. $\alpha = \alpha_{t_+}$.
2. $\alpha = \alpha_{v_-}$.

From Fig. 5.2, what are m and p in each case? The relevant data is in Eqs. (5.1.8) and (4.3.7). In the latter, $\alpha_{1,2,3} \to \{\alpha_{t_+}, \alpha_{u_+}, \alpha_{v_+}\}$.

• **Exercise**: Verify that Eq. (5.1.23) holds for $M = \alpha_{u_-}$ in the adjoint representation of su(3), for three cases:

[1] Georgi [3] refers to it as the "master formula," while others call it the depth equation.

1. $\alpha = \alpha_{v_+}$.

2. $\alpha = \alpha_{t_-}$.

3. $\alpha = \alpha_{u_+}$.

From Fig. 5.3, what are m and p in each case? The relevant data is in Eq. (4.3.7). In the latter, $\alpha_{1,2,3} \to \{\alpha_{t_+}, \alpha_{u_+}, \alpha_{v_+}\}$.

5.2 Root geometries

In this section, we will state and/or prove various important facts about root systems (= adjoint representations). The goal is to generalize beyond su(3) to generic semi-simple Lie algebras.

5.2.1 Weyl reflections

Consider a weight chain with $q+1$ weights such that q is even. Since the number of weights is odd, there is a particular weight at the center of the chain; denote it as M_0. Obviously $m = p = q/2$ for M_0, where m (p) denotes the number of weights below (above) it. We can denote a generic weight in the chain as

$$M = M_0 + n\alpha, \qquad 0 \le |n| \le q/2, \tag{5.2.1}$$

where $n = (m-p)/2$ and there are m weights below (p weights above) M. Consider a **Weyl reflection** about the chain center,

$$M \to M_0 - n\alpha = M - 2n\alpha = M - (m-p)\alpha \tag{5.2.2}$$

so that

$$M \to M - \langle M, \alpha^\vee \rangle\, \alpha, \qquad \text{Weyl reflection } \equiv S_\alpha. \tag{5.2.3}$$

• **Exercise**: Verify that Eq. (5.2.3) also holds for a reflection about the center of an α-weight chain with $q+1$ weights, with q odd.

For each positive root α we can define an associated Weyl reflection operation S_α. With respect to the su(2) generated by $\{e_{-\alpha}, h_\alpha, e_\alpha\}$ [Eq. (5.1.1)], the adjoint representation decomposes into "parallel" strings of weights of varying lengths. Applying the Weyl reflection interchanges the weights along these strings. **We assert that a valid root system is isometric under all Weyl reflections, i.e. the "picture" of the root system is invariant.** This is illustrated for su(3) in Fig. 5.4. Note that a weight that is "orthogonal" to α is left invariant according to Eq. (5.2.3).

The product of all Weyl reflections forms the **Weyl group**, which is a subgroup of the isometry group for the root system.[2] We will examine the Weyl group in more detail when we derive Weyl's character formula in module 8.

[2] Note however that the Weyl group itself contains more than Weyl reflections, since the product of two reflections is equivalent to a rotation. The Weyl group for any root system can be analyzed with finite group theory.

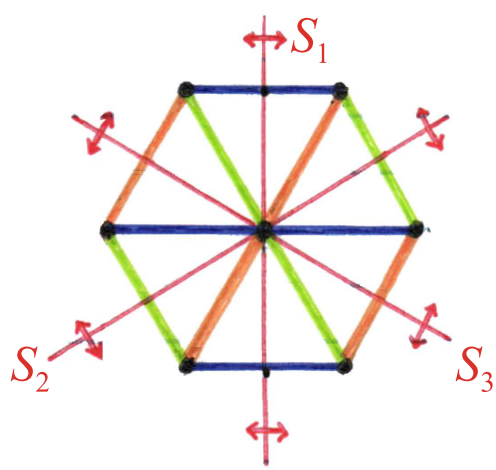

Figure 5.4: Weyl reflections $S_{1,2,3}$ in the su(3) root system (adjoint representation). Reflection S_i is associated with positive root α_i [Eq. (5.2.3)], and $\alpha_{1,2,3} \leftrightarrow \alpha_{t_+,u_+,v_+}$ (c.f. Figs. 4.1 and 5.3).

5.2.2 Root chains and inner products

We assert the following facts about roots, some of which will be proven below.

I. If α is a root, then

$$\langle \alpha, \alpha \rangle \equiv |\alpha|^2 = (h_\alpha, h_\alpha) > 0. \tag{5.2.4}$$

II. If α is a root, so is $-\alpha$. These are associated to the "raising" and "lowering" root vector operators $e_{\pm\alpha}$.

III. The *only* allowed multiples of α in the root system are $\{-\alpha, 0, \alpha\}$, with the center weight belonging to the Cartan subalgebra H.

IV. Root systems come in two different varieties. Either (a) all roots have the same norm $|\alpha|^2$, or (b) each root has one of two lengths, "short" or "long." A Lie algebra in which all roots have the same length is said to be **simply-laced**.

V. If $\{\alpha, \beta, \alpha+\beta\}$ are all roots, then

$$[e_\alpha, e_\beta] = N_{\alpha,\beta}\, e_{\alpha+\beta}, \qquad N_{\alpha,\beta} \neq 0. \tag{5.2.5}$$

In other words, "all possible moves are allowed in the root system."

VI. There is only one linearly independent root vector associated to each root. The **weight space** is defined as the degeneracy of a particular weight in an irreducible representation. The weight space of all weights in the adjoint is one-dimensional, except for the r-dimensional Cartan subalgebra with weight 0.

VII. In the adjoint representation, the maximum length of any weight chain is 4.

VIII. If α and β are roots, then $\langle \alpha, \beta \rangle \in \mathbb{Q}$, where $\mathbb{Q}$ denotes the field of rational numbers. The set of roots forms an overcomplete basis for H^*, but we can choose a subset $\{\overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_r\}$ to use as a complete basis. Then any weight ρ can be expanded as

$$\rho = \sum_{j=1}^{r} \kappa_j^{(\rho)}\, \overline{\alpha}_j, \quad \kappa_j^{(\rho)} \in \mathbb{Q}. \tag{5.2.6}$$

We comment on or prove some of these statements (I.)–(VIII.) as follows.

(II.) **Proof**: Recall that

$$(e_\alpha, e_\beta) = 0$$

unless $\alpha + \beta = 0$ [Eq. (4.4.5)]. If $-\alpha$ is not a root, then

$$(e_\alpha, x) = 0 \;\; \forall\, x \in L.$$

However, a semi-simple Lie algebra has a non-degenerate Killing form, so that the determinant of the "overlap matrix" of all Cartan-Weyl basis elements (x_i, x_j) cannot vanish. Therefore $-\alpha$ must be a root if α is.

(III.) This establishes the generalization of the picture we had for su(3): a generic root system is a set of su(2)s $\{-\alpha_i, 0, \alpha_i\}$ that can be drawn as "spokes" passing through a central hub (the Cartan subalgebra H). The root geometry can be embedded in r Euclidean dimensions. The point group symmetry is implied by the root inner products $\{\langle \alpha_i, \alpha_j \rangle\}$. Roots can have one or two different norms $|\alpha|^2$, allowing us to distinguish "short" and "long" roots (IV.). In **simply-laced** root systems like su(3), all roots have the same length. An example with short and long roots is the G_2 root system, shown in Fig. 5.5.

(V.) This will be proven in homework, using the MWDF [Eq. (5.1.23)].

(VII.) **Proof:** Assume that a string with 5 weights exists in the adjoint representation. WLoG we can write this as

$$\{\beta - 2\alpha, \beta - \alpha, \beta, \beta + \alpha, \beta + 2\alpha\}.$$

Now, by assertion [(III.)]

$$2\alpha = (\beta + 2\alpha) - \beta \text{ is } \underline{\text{not}} \text{ a root, nor is}$$
$$2(\beta + \alpha) = (\beta + 2\alpha) + \beta.$$

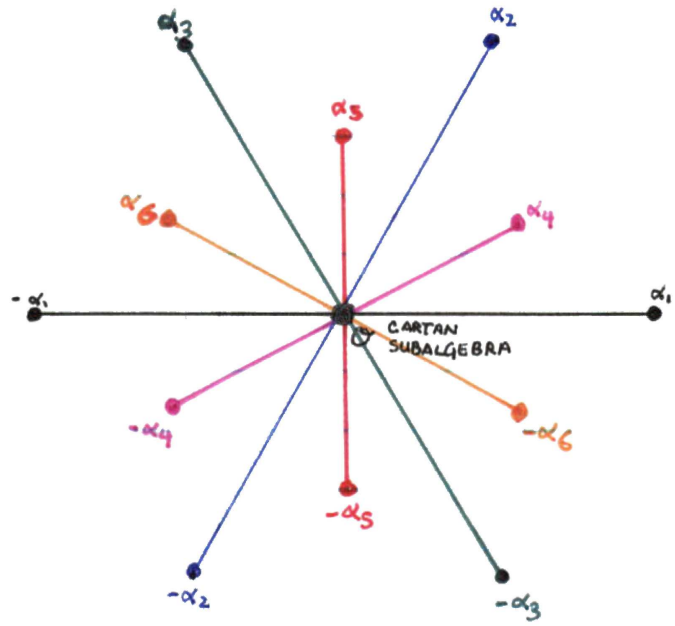

Figure 5.5: The root system for the rank-two exceptional Lie algebra G_2.

These conditions imply that $\beta + 2\alpha$ is a string of length 1 with respect to the actions of $\hat{E}_{\pm\beta}$, i.e. $ad_{e_{\pm\beta}}(e_{\beta+2\alpha}) = 0$. Similarly

$$\begin{aligned} -2\alpha &= (\beta - 2\alpha) - \beta \text{ is } \underline{\text{not}} \text{ a root, nor is} \\ 2(\beta - \alpha) &= (\beta - 2\alpha) + \beta, \end{aligned}$$

so that $\beta - 2\alpha$ is a string of length 1 with respect to the actions of $\hat{E}_{\pm\beta}$. Eq. (5.1.23) then implies that

$$(m - p)_{\beta+2\alpha;\beta} = 0 = \langle \beta + 2\alpha, \beta^\vee \rangle, \qquad (m' - p')_{\beta-2\alpha;\beta} = 0 = \langle \beta - 2\alpha, \beta^\vee \rangle, \tag{5.2.7}$$

where the subscripts on the left-hand-sides mean that m and p are determined for the weights $\beta \pm 2\alpha$ in β-strings. Combining these equations gives

$$\frac{\langle \beta, \beta \rangle}{\langle \beta, \beta \rangle} = 0,$$

which is inconsistent [(I.)]. Q.E.D.

(VIII.) This will be proven in homework, using the MWDF [Eq. (5.1.23)].

Further information about root geometry can be extracted from the MWDF Eq. (5.1.23). Consider

$$(m - p)_{\alpha;\beta} = \langle \alpha, \beta^\vee \rangle \equiv M, \qquad (m' - p')_{\beta;\alpha} = \langle \beta, \alpha^\vee \rangle \equiv N, \tag{5.2.8}$$

where again the subscript $\{\alpha; \beta\}$ means that m and p are determined for weight α in a string generated by $e_{\pm\beta}$. In Eq. (5.2.8), M and N are some integers. Then we have

$$\frac{MN}{4} = \frac{\langle \alpha, \beta \rangle^2}{|\alpha|^2 |\beta|^2}, \qquad |\alpha|^2 \equiv \langle \alpha, \alpha \rangle, \tag{5.2.9}$$

so that

$$\frac{MN}{4} \equiv \cos^2(\theta) \leq 1, \tag{5.2.10}$$

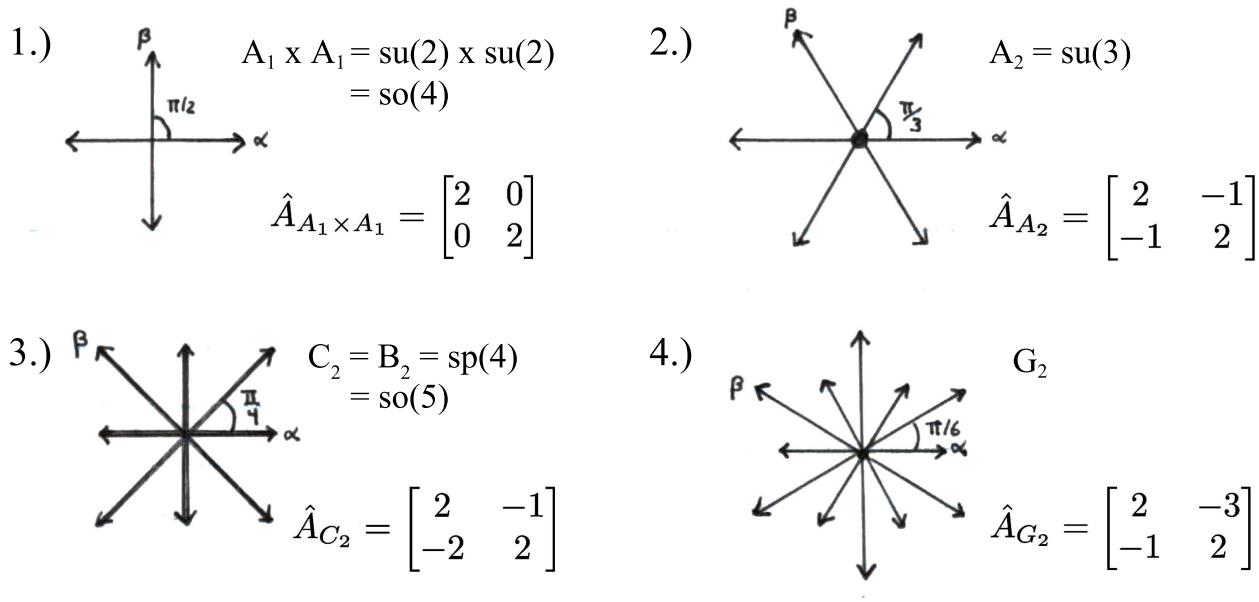

Figure 5.6: The four rank-two root systems. The Cartan matrix $\hat{A}$ is given for each; it is defined in Sec. 5.3.1, below.

and equal to one only for $\beta = \pm\alpha$. Given that a weight string can have at most 4 weights in the adjoint representation, it implies that

$$|M|, |N| \in \{0, 1, 2, 3\} \quad \Rightarrow \quad |M||N| \in \{0, 1, 2, 3, 4, 6, 9\},$$

where the possibilities in red are disallowed because of Eq. (5.2.10). We therefore conclude that the only allowed values for $\cos\theta$ in Eq. (5.2.10) are

$$\cos\theta \in \left\{0, \pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{\sqrt{3}}{2}, \pm 1\right\} \quad \Leftrightarrow \quad \theta \in \left\{0, \pi; \frac{\pi}{6}, \frac{5\pi}{6}; \frac{\pi}{4}, \frac{3\pi}{4}; \frac{\pi}{3}, \frac{2\pi}{3}; \frac{\pi}{2}\right\}. \tag{5.2.11}$$

It turns out that this severely restricts the possibilities for root geometries. In the next subsection, we will obtain further constraints by considering a basis for H^*. For now, we claim that the full list of root systems is exhausted by four "classical" families of algebras, and 5 particular "exceptional" algebras. The classical algebra families are

$$\begin{aligned} A_n &= \mathrm{su}(n+1), & n \geq 1, \\ B_n &= \mathrm{so}(2n+1), & n \geq 1, \\ C_n &= \mathrm{sp}(2n), & n \geq 1, \\ D_n &= \mathrm{so}(2n), & n \geq 2. \end{aligned} \tag{5.2.12}$$

Here the subscript n indicates the rank. The exceptional algebras are

$$G_2, \quad F_4, \quad E_6, \quad E_7, \quad E_8. \tag{5.2.13}$$

The classical families $\{A_n, B_n, C_n, D_n\}$ routinely crop up as symmetries in quantum mechanics and quantum field theory. The exceptional algebras are more exotic, but have found applications e.g. in string theory and in the theory of massive integrable 1+1-D quantum field theories.

We can also organize algebras by rank r. For small r, there are numerous "accidental" equivalences between members of the classical families. For example, there is only one unique algebra at $r = 1$, $\mathrm{su}(2) = \mathrm{so}(3) = \mathrm{sp}(2)$ ($A_1 = B_1 = C_1$). The four rank-two root systems are shown in Fig. 5.6. As discussed in module 2, $\mathrm{so}(4) = \mathrm{su}(2) \times \mathrm{su}(2)$, so that $D_2 = A_1 \times A_1$. $A_2 = \mathrm{su}(3)$ is the second example. The third is $B_2 = C_2$, which corresponds to the accidental equivalence $\mathrm{so}(5) = \mathrm{sp}(4)$. As we will show later, in this case the defining representation of $\mathrm{sp}(4)$ is the **spinor representation** of $\mathrm{so}(5)$, similar to the half-integer representations of $\mathrm{su}(2) = \mathrm{so}(3)$. The final rank-2 root system is that of the exceptional algebra G_2.

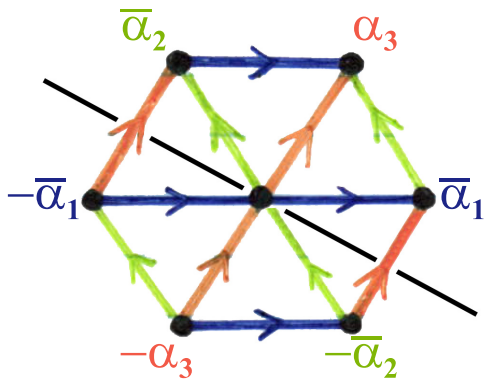

Figure 5.7: Positive and simple roots for su(3). Relative to Fig. 5.3 we relabel $\{t_\pm, u_\pm, v_\pm\} \Rightarrow \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3\}$. The simple roots are $\overline{\alpha}_{1,2}$. The positive roots are $\{\overline{\alpha}_1, \overline{\alpha}_2, \alpha_3\}$, where $\alpha_3 = \overline{\alpha}_1 + \overline{\alpha}_2 \equiv \theta$, the highest root.

5.3 Constructing the adjoint representation using the MWDF

5.3.1 Simple roots and the Cartan matrix

We can construct the adjoint representation for any Lie algebra given some minimal data about the root system. The needed information concerns the **simple roots**, which are defined as follows.

For some Lie algebra L, choose a basis in H_0^* (the space of real combinations of the roots) as a subset of the roots $\{\tilde{\alpha}_i\}$, $i \in \{1, 2, \ldots, r\}$; r is the rank. Any Cartan-Weyl basis weight (root or H-basis element) β can be expanded as [Proposition (VIII.), Sec. 5.2.2]

$$\beta = \sum_{i=1}^{r} q_i \, \tilde{\alpha}_i, \quad q_i \in \mathbb{Q}. \tag{5.3.1}$$

Let us denote the complete set of roots as Δ and the total number of roots by $|\Delta|$; $|\Delta| + r = d$, the dimension of the Lie group. We introduce "lexicographic ordering" on Δ:

- β is a **positive root** if $q_1 > 0$, *or* $q_1 = 0$ and $q_2 > 0$, *or* $q_1 = q_2 = 0$, and $q_3 > 0$, $\ldots$, *or* $q_1 = q_2 = \cdots = q_{r-1} = 0$ and $q_r > 0$. Denote the set of positive roots as Δ_+; for $\beta \in \Delta_+$, we write $\beta > 0$. By definition, all basis roots $\{\tilde{\alpha}_i\}$ are positive.
- If $\Delta_+ \equiv \{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ denotes the set of positive roots, then the set of negative roots Δ_- is defined as $\{-\alpha_1, -\alpha_2, \ldots, -\alpha_p\}$. Let $|\Delta_\pm|$ denote the number of positive or negative roots, so that $|\Delta_+| = |\Delta_-| = |\Delta|/2$.
- If α and β are roots, then $\alpha > \beta$ if $\alpha - \beta$ is positive.

Note that a Cartan subalgebra basis element has $q_i = 0$ for all $i \in \{1, \ldots, r\}$.

This naturally (if arbitrarily) orders the set of all roots. In particular, there will be a **highest root** θ that satisfies $\theta > \alpha$ for all roots $\alpha \neq \theta$. The highest root cannot be "raised" by acting with any positive root vector,

$$ad_{e_\alpha}(e_\theta) = [e_\alpha, e_\theta] = 0, \quad \alpha \in \Delta_+. \tag{5.3.2}$$

A **simple root** is denoted with a bar $\overline{\rho}$, and satisfies two conditions:

1. It is positive $\overline{\rho} \in \Delta_+$,
2. It cannot be written as a sum of positive roots, $\overline{\rho} \neq \alpha + \beta$ for $\alpha, \beta \in \Delta_+$.

• su(3) example: The root system is shown in Fig. 5.7, where relative to Fig. 5.3 we have relabeled $\{t_\pm, u_\pm, v_\pm\} \Rightarrow \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3\}$. The black line divides the algebra into positive and negative roots. Positive roots $\overline{\alpha}_1$ and $\overline{\alpha}_2$ are also simple, but $\alpha_3 = \overline{\alpha}_1 + \overline{\alpha}_2$ is not. In fact, $\alpha_3 = \theta$, the highest root.

Let Π denote the set of simple roots. Let $\overline{\alpha}, \overline{\beta} \in \Pi$ ($\overline{\alpha} \neq \overline{\beta}$). We assert the following simple root facts:

I. $\overline{\alpha} - \overline{\beta}$ and $\overline{\beta} - \overline{\alpha}$ are not roots, which implies that [Eq. (4.4.11c)]

$$a_{d_{e_{-\overline{\beta}}}}(e_{\overline{\alpha}}) = [e_{-\overline{\beta}}, e_{\overline{\alpha}}] = 0, \quad a_{d_{e_{-\overline{\alpha}}}}(e_{\overline{\beta}}) = [e_{-\overline{\alpha}}, e_{\overline{\beta}}] = 0. \tag{5.3.3}$$

In other words, simple roots are lowest weight states in mutually-generated strings.

Proof: Either $\overline{\alpha} - \overline{\beta}$ or $\overline{\beta} - \overline{\alpha}$ is positive. Then either $\overline{\alpha} = (\overline{\alpha} - \overline{\beta}) + \overline{\beta}$ or $\overline{\beta} = (\overline{\beta} - \overline{\alpha}) + \overline{\alpha}$ is not simple (contradiction).

II. $\langle \overline{\alpha}, \overline{\beta} \rangle \leq 0$, i.e. the angle between simple roots is obtuse (or 90°).

Proof: The MWDF Eq. (5.1.23) implies that

$$(m-p)_{\overline{\alpha};\overline{\beta}} = -p_{\overline{\alpha};\overline{\beta}} = \frac{2\langle \overline{\alpha}, \overline{\beta} \rangle}{\langle \overline{\beta}, \overline{\beta} \rangle} \quad \Rightarrow \quad \langle \overline{\alpha}, \overline{\beta} \rangle \leq 0, \tag{5.3.4}$$

since $\langle \overline{\beta}, \overline{\beta} \rangle > 0$ [Proposition (I.), Sec. 5.2.2].

III. The set of simple roots Π is linearly independent, and can be employed as a basis for the positive roots.

Proof: Assume that this is not true, and write $\Pi = \Pi_1 \oplus \Pi_2$. Linear dependence means

$$\sum_{\overline{\alpha}_i \in \Pi_1} a_i \, \overline{\alpha}_i - \sum_{\overline{\beta}_j \in \Pi_2} b_j \, \overline{\beta}_j = 0, \quad a_i, b_j \geq 0 \ \ \forall \ \ \{i, j\}. \tag{5.3.5}$$

Take the inner product with respect to the first term to get

$$\sum_{\overline{\alpha}_i, \overline{\alpha}_j \in \Pi_1} a_i a_j \, \langle \overline{\alpha}_i, \overline{\alpha}_j \rangle = \sum_{\overline{\alpha}_i \in \Pi_1, \overline{\beta}_j \in \Pi_2} a_i b_j \, \langle \overline{\alpha}_i, \overline{\beta}_j \rangle. \tag{5.3.6}$$

The left-hand side is a positive squared-norm, but the right-hand side is a sum of negative terms (contradiction).

IV. Every positive root $\rho > 0$ can be written as a sum of simple roots with non-negative, integer coefficients $\{\kappa_i^{(\rho)}\}$:

$$\begin{aligned} &\rho = \sum_{i=1}^{r} \kappa_i^{(\rho)} \, \overline{\alpha}_i, \qquad \rho \in \Delta_+, \ \ \kappa_i^{(\rho)} \in \mathbb{N}_0, && \text{Decomposition of a positive root in terms of simple roots with non-negative integer coefficients.} \\ &k \equiv \sum_{i=1}^{r} \kappa_i^{(\rho)} \in \{1, 2, \ldots\}, && \textbf{Level } k \text{ of the positive root } \rho. \end{aligned} \tag{5.3.7}$$

Here $\mathbb{N}_0$ denotes the set of natural numbers including zero.

Proof: Eq. (5.3.7) trivially holds if ρ itself is simple; the level $k = 1$ in this case. If ρ is not simple ($k \geq 2$), it can be expressed as a sum of two positive roots, $\rho = \lambda + \gamma$, $\{\lambda, \gamma\} \in \Delta_+$. Either of these are simple, or can be further decomposed. Continue this bipartitioning process until all components are simple roots, and this gives the explicit decomposition of ρ.

In order to construct higher level roots from simple roots, it is extremely useful to introduce the **Cartan matrix** A_{ij},

$$\boxed{A_{ij} \equiv \langle \overline{\alpha}_i, \overline{\alpha}_j^{\vee} \rangle = \frac{2\langle \overline{\alpha}_i, \overline{\alpha}_j \rangle}{\langle \overline{\alpha}_j, \overline{\alpha}_j \rangle}, \qquad \text{Cartan matrix.}} \tag{5.3.8}$$

We emphasize that the Cartan matrix is generally *not* symmetric. The Cartan matrix encodes two different types of information: (1) the simple root geometry (angles between and relative lengths of simple roots), and (2) via the MWDF Eq. (5.1.23), the "height" of positive weight strings, starting from a simple root. In particular

Attributes of the Cartan matrix

1. The matrix elements of A_{ij} are integers. For $i \neq j$ the elements are non-positive integers

$$A_{ij} = -p_{\overline{\alpha}_i;\overline{\alpha}_j}, \tag{5.3.9}$$

where $p_{\overline{\alpha}_i;\overline{\alpha}_j} + 1$ is the length of the weight string generated by "raising" $\overline{\alpha}_i$ with the root vector $e_{\overline{\alpha}_j}$: $\{\overline{\alpha}_i, \overline{\alpha}_i + \overline{\alpha}_j, \ldots \overline{\alpha}_i + (p_{\overline{\alpha}_i;\overline{\alpha}_j})\,\overline{\alpha}_j\}$.

From Proposition (VII.) in Sec. 5.2.2, we conclude that

$$-3 \leq A_{ij} \leq 0, \quad i \neq j. \tag{5.3.10}$$

In particular, $A_{ij} = 0$ for $i \neq j$ implies that $\overline{\alpha}_i + \overline{\alpha}_j$ is not a root, so that

$$A_{ij} = 0 \quad \Leftrightarrow \quad \langle \overline{\alpha}_i, \overline{\alpha}_j \rangle = 0 \quad \Leftrightarrow \quad [e_{\overline{\alpha}_i}, e_{\overline{\alpha}_j}] = 0, \quad i \neq j. \tag{5.3.11}$$

• Orthogonal simple roots correspond to a vanishing root vector commutator.

Each diagonal element $A_{ii} = 2$ (no summation on i), since this is the depth of the su(2) subalgebra generated by lowering with $e_{-\overline{\alpha}_i}$,

$$A_{ii} = m_{\overline{\alpha}_i;\overline{\alpha}_i} = 2.$$

2. The angle between simple roots $\overline{\alpha}_i$ and $\overline{\alpha}_j$ is encoded in the product

$$\frac{A_{ij}A_{ji}}{4} = \frac{\langle \overline{\alpha}_i, \overline{\alpha}_j \rangle^2}{|\overline{\alpha}_i|^2 |\overline{\alpha}_j|^2} = \cos^2\theta \leq 1, \tag{5.3.12}$$

[c.f. Eq. (5.2.8)–(5.2.10)]. Combining Eq. (5.2.11) and Proposition (II.) above, we know that

$$\theta \in \{90°, 120°, 135°, 150°\} \text{ for } i \neq j \quad \Rightarrow \quad A_{ij} \in \{0, -1, -2, -3\}, \text{ for } i \neq j, \tag{5.3.13}$$

consistent with Eq. (5.3.10).

3. The relative squared lengths of simple roots $\overline{\alpha}_i$ and $\overline{\alpha}_j$ are encoded in the ratio

$$\frac{A_{ij}}{A_{ji}} = \frac{|\overline{\alpha}_i|^2}{|\overline{\alpha}_j|^2}. \tag{5.3.14}$$

4. $\det \hat{A} \neq 0$, since the simple roots form a basis for H_0^*.

• su(3) example: The simple roots are $\overline{\alpha}_1$ and $\overline{\alpha}_2$, see Fig. 5.7. From Eq. (4.3.7),

$$\langle \overline{\alpha}_1, \overline{\alpha}_1 \rangle = \langle \overline{\alpha}_2, \overline{\alpha}_2 \rangle = 1/3, \quad \langle \overline{\alpha}_1, \overline{\alpha}_2 \rangle = -1/6,$$

leading to the Cartan matrix

$$\hat{A}_{A_2} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}. \tag{5.3.15}$$

Cartan's notation A_n for su(n+1) was introduced in Eq. (5.2.12).

5.3.2 Higher level roots

We can iteratively construct the complete set of positive roots from the simple ones. A generic positive root $\rho \in \Delta_+$ can be expanded as in Eq. (5.3.7). Using the MWDF Eq. (5.1.23), we rewrite this equation as

$$p_{\rho;\overline{\alpha}_i} = m_{\rho;\overline{\alpha}_i} - \sum_{j=1}^{r} \kappa_j^{(\rho)} A_{ji}, \tag{5.3.16}$$

where A_{ji} is the Cartan matrix, and $p_{\rho;\overline{\alpha}_i}$ ($m_{\rho;\overline{\alpha}_i}$) denotes the number of weights above (below) ρ in the chain generated by $e_{\pm\overline{\alpha}_i}$.

The higher root construction algorithm now proceeds as follows.

Root building algorithm

1. At level $k=1$ we have the simple roots $\{\overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_r\}$. Let $\rho = \overline{\alpha}_{j_0}$ (a particular simple root), so that

$$p_{\rho;\overline{\alpha}_i} = -A_{j_0\, i}, \quad (i \neq j_0). \tag{5.3.17}$$

$\Rightarrow$ The number of weights above ρ in the chain generated by raising with $e_{\overline{\alpha}_i}$ is given by $-A_{j_0\, i}$. We label each simple root $\rho = \overline{\alpha}_{j_0}$ by a <u>row</u> of the Cartan matrix

$$[A_{j_0\,1}\, A_{j_0\,2} \cdots A_{j_0\,r}] \equiv [\Lambda_1^{(1)}\, \Lambda_2^{(1)} \cdots \Lambda_r^{(1)}]. \tag{5.3.18}$$

The integers $\{\Lambda_i^{(1)}\}$ are the **Dynkin labels** for the level-1 weight $\rho = \overline{\alpha}_{j_0}$.

- <u>su(3)</u>: Using Eq. (5.3.15), the Dynkin labels of the simple roots are

$$\underset{(\overline{\alpha}_1)}{\begin{bmatrix} 2 & -1 \end{bmatrix}} \quad \underset{(\overline{\alpha}_2)}{\begin{bmatrix} -1 & 2 \end{bmatrix}} \tag{5.3.19}$$

2. To find all level $k=2$ roots, examine the Dynkin labels $[\Lambda_1 \cdots \Lambda_r]$ for each simple root $\overline{\alpha}_{j_0}$. Whenever $A_{j_0\, j_1} < 0$, $\rho = \overline{\alpha}_{j_0} + \overline{\alpha}_{j_1}$ is a valid positive root. For each valid positive root ρ at $k=2$, record the sum of the corresponding rows of the Cartan matrix,

$$\sum_{j=1}^{r} \kappa_j^{(\rho)} A_{ji} = A_{j_0\, i} + A_{j_1\, i} \quad \Leftrightarrow \quad [(A_{j_0\,1} + A_{j_1\,1}) \cdots (A_{j_0\,r} + A_{j_1\,r})] \equiv [\Lambda_1^{(2)}\, \Lambda_2^{(2)} \cdots \Lambda_r^{(2)}]. \tag{5.3.20}$$

- <u>su(3)</u>: There is one root α_3 at level 2 with Dynkin coefficients

$$\underset{(\overline{\alpha}_1 + \overline{\alpha}_2)}{\begin{bmatrix} 1 & 1 \end{bmatrix}} \tag{5.3.21}$$

Each $k=2$ root can arise in multiple ways, e.g. by raising $\overline{\alpha}_{j_0}$ with $e_{\overline{\alpha}_{j_1}}$ or by raising $\overline{\alpha}_{j_1}$ with $e_{\overline{\alpha}_{j_0}}$. For each $k=2$ root ρ, record the set of non-zero $\{m_{\rho;\overline{\alpha}_i}\}$ ($m_{\rho;\overline{\alpha}_i} \in \{0,1\}$, $i \in \{1,\ldots,r\}$), which specify whether a level-1 simple root can be accessed by acting with each $e_{-\overline{\alpha}_i}$ on ρ.

- <u>su(3)</u>:

$$\rho = \alpha_3: \quad m_{\alpha_3;\overline{\alpha}_1} = m_{\alpha_3;\overline{\alpha}_2} = 1. \tag{5.3.22}$$

3. To find all level $k=3$ roots, proceed as follows. Consider each $k=2$ root ρ. To determine if $\rho + \overline{\alpha}_i$ is a valid root, compute

$$p_{\rho;\overline{\alpha}_i} = m_{\rho;\overline{\alpha}_i} - \Lambda_i^{(2)}, \tag{5.3.23}$$

where $m_{\rho;\overline{\alpha}_i} \in \{0,1\}$ and the Dynkin coefficient $\Lambda_i^{(2)}$ were computed in step (2.), above. If $p_{\rho;\overline{\alpha}_i} > 0$, then $\rho + \overline{\alpha}_i$ is a valid root.

- <u>su(3)</u>: The only $k=2$ root is α_3. We have

$$\begin{aligned} p_{\alpha_3;\overline{\alpha}_1} &= m_{\alpha_3;\overline{\alpha}_1} - \Lambda_1^{(2)} = 1 - 1 = 0, \\ p_{\alpha_3;\overline{\alpha}_2} &= m_{\alpha_3;\overline{\alpha}_2} - \Lambda_2^{(2)} = 1 - 1 = 0. \end{aligned} \tag{5.3.24}$$

We conclude that $\alpha_3 = \theta$, the highest root in su(3). This algebra terminates at level $k=2$.

For each $k=3$ root ρ, record the set of non-zero $\{m_{\rho;\overline{\alpha}_i}\}$ ($m_{\rho;\overline{\alpha}_i} \in \{0,1,2\}$, $i \in \{1,\ldots,r\}$), which specify the number of lower-level roots below ρ accessed by acting repeatedly with each $e_{-\overline{\alpha}_i}$ on ρ.

The algorithm iterates until the highest root θ is reached.

5.3.3 G_2 example

Let us illustrate the algorithm with a less trivial rank-2 example: the exceptional Lie algebra G_2. The Cartan matrix is (as quoted in Fig. 5.6)

$$\hat{A}_{G_2} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}. \tag{5.3.25}$$

1. At $k = 1$, we have the simple roots

$$\underset{(\overline{\alpha}_1)}{\begin{bmatrix} 2 & -3 \end{bmatrix}} \quad \underset{(\overline{\alpha}_2)}{\begin{bmatrix} -1 & 2 \end{bmatrix}} \tag{5.3.26a}$$

2. At $k = 2$, the sum $\rho^{(2)} \equiv \overline{\alpha}_1 + \overline{\alpha}_2$ is a valid root since [Eq. (5.3.17)]

$$\begin{aligned} p_{\overline{\alpha}_2;\overline{\alpha}_1} &= 1, \\ p_{\overline{\alpha}_1;\overline{\alpha}_2} &= 3. \end{aligned} \tag{5.3.26b}$$

The Dynkin labels obtain from summing those of $\overline{\alpha}_{1,2}$,

$$\underset{(\overline{\alpha}_1+\overline{\alpha}_2)}{\begin{bmatrix} 1 & -1 \end{bmatrix}} \tag{5.3.26c}$$

The depths are

$$m_{\rho^{(2)};\overline{\alpha}_1} = m_{\rho^{(2)};\overline{\alpha}_2} = 1. \tag{5.3.26d}$$

3. At $k = 3$, we compute

$$\begin{aligned} p_{\rho^{(2)};\overline{\alpha}_1} &= m_{\rho^{(2)};\overline{\alpha}_1} - 1 = 0, \\ p_{\rho^{(2)};\overline{\alpha}_2} &= m_{\rho^{(2)};\overline{\alpha}_2} + 1 = 2. \end{aligned} \tag{5.3.26e}$$

$\rho^{(3)} = \rho^{(2)} + \overline{\alpha}_2$ is the only $k = 3$ root,

$$\underset{(\overline{\alpha}_1+2\overline{\alpha}_2)}{\begin{bmatrix} 0 & 1 \end{bmatrix}} \tag{5.3.26f}$$

The depths are

$$m_{\rho^{(3)};\overline{\alpha}_1} = 0, \qquad m_{\rho^{(3)};\overline{\alpha}_2} = 2. \tag{5.3.26g}$$

4. At $k = 4$,

$$\begin{aligned} p_{\rho^{(3)};\overline{\alpha}_1} &= m_{\rho^{(3)};\overline{\alpha}_1} - 0 = 0, \\ p_{\rho^{(3)};\overline{\alpha}_2} &= m_{\rho^{(3)};\overline{\alpha}_2} - 1 = 1. \end{aligned} \tag{5.3.26h}$$

$\rho^{(4)} = \rho^{(3)} + \overline{\alpha}_2$ is the only $k = 4$ root,

$$\underset{(\overline{\alpha}_1+3\overline{\alpha}_2)}{\begin{bmatrix} -1 & 3 \end{bmatrix}} \tag{5.3.26i}$$

The depths are

$$m_{\rho^{(4)};\overline{\alpha}_1} = 0, \qquad m_{\rho^{(4)};\overline{\alpha}_2} = 3. \tag{5.3.26j}$$

5. At $k = 5$,

$$\begin{aligned} p_{\rho^{(4)};\overline{\alpha}_1} &= m_{\rho^{(4)};\overline{\alpha}_1} + 1 = 1, \\ p_{\rho^{(4)};\overline{\alpha}_2} &= m_{\rho^{(4)};\overline{\alpha}_2} - 3 = 0. \end{aligned} \tag{5.3.26k}$$

$\rho^{(5)} = \rho^{(4)} + \overline{\alpha}_1$ is the only $k = 5$ root,

$$\underset{(2\overline{\alpha}_1+3\overline{\alpha}_2)}{\begin{bmatrix} 1 & 0 \end{bmatrix}} \tag{5.3.26l}$$

The depths are

$$m_{\rho^{(5)};\overline{\alpha}_1} = 1, \qquad m_{\rho^{(5)};\overline{\alpha}_2} = 0. \tag{5.3.26m}$$

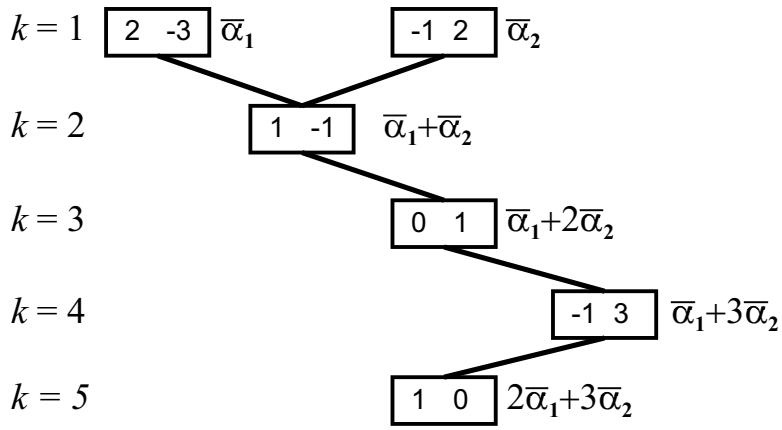

Figure 5.8: Positive roots at successive levels k in G_2. For a rank-2 algebra such as this, we can graphically distinguish "left-moves" (raising by $e_{\overline{\alpha}_1}$) from "right-moves" (raising by $e_{\overline{\alpha}_2}$). Branching diagrams can be constructed for the adjoint representation of any Lie algebra, although planar figures become cumbersome for higher rank algebras.

6. At $k = 6$,

$$\begin{aligned} p_{\rho^{(5)};\overline{\alpha}_1} &= m_{\rho^{(5)};\overline{\alpha}_1} - 1 = 0, \\ p_{\rho^{(5)};\overline{\alpha}_2} &= m_{\rho^{(5)};\overline{\alpha}_2} - 0 = 0. \end{aligned} \tag{5.3.26n}$$

We conclude that there are no $k = 6$ roots. $\rho^{(5)} = 2\overline{\alpha}_1 + 3\overline{\alpha}_2 \equiv \theta$ is the highest root.

The results of the positive-root-finding algorithm can be indicated with a branching diagram, shown in Fig. 5.8. Since the roots are non-degenerate [Proposition (VI.), Sec. 5.2.2] and each positive root ρ has a corresponding negative root $-\rho$, we have a complete inventory of the roots for G_2. There are $|\Delta| = 12$ roots, so the group dimension $d = 12 + 2 = 14$.

To get a geometrical picture of the algebra, we examine the Cartan matrix in Eq. (5.3.25). The angle between simple roots $\overline{\alpha}_1$ and $\overline{\alpha}_2$ is encoded in the product [Eq. (5.3.12)]

$$\frac{A_{12}A_{21}}{4} = \cos^2\theta = \frac{3}{4} \quad \Rightarrow \quad \theta = 150^\circ. \tag{5.3.27}$$

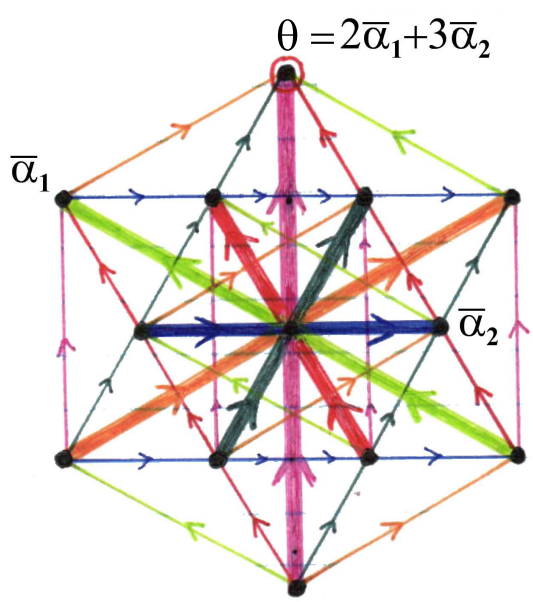

Figure 5.9: The adjoint representation of the exceptional Lie algebra G_2.

$$\hat{A}_{A_1\times A_1} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\hat{A}_{A_2} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\hat{A}_{C_2} = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$$

$$\hat{A}_{G_2} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

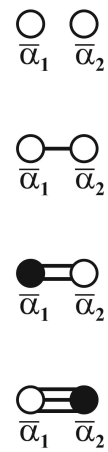

Figure 5.10: The Dynkin diagrams for rank-2 algebras.

The relative squared lengths of simple roots $\overline{\alpha}_1$ and $\overline{\alpha}_2$ are encoded in the ratio [Eq. (5.3.14)]

$$\frac{A_{12}}{A_{21}} = \frac{|\overline{\alpha}_1|^2}{|\overline{\alpha}_2|^2} = 3, \tag{5.3.28}$$

so that $|\overline{\alpha}_1| = \sqrt{3}|\overline{\alpha}_2|$. Combining this information with the root inventory, we immediately get the picture shown in Fig. 5.9. We note that the roots come in two varieties, short and long, and that the highest root θ is a long root. For non-simply-laced algebras with short and long roots, it turns out that the highest root is always long.

5.3.4 Dynkin diagrams

The Cartan matrix A_{ij} [Eq. (5.3.8)] succinctly encodes the entire structure of a Lie algebra. One can use Dynkin coefficients $\begin{bmatrix}\Lambda_1 & \Lambda_2 & \cdots & \Lambda_r\end{bmatrix}$ to define all possible highest weight states in finite-dimensional irreducible representations. Then one employs a variant of the root-building algorithm presented above to determine all weights in the representation (module 7). Weight space multiplicities obtain via Freudenthal's formula (module 8). All commutation relations can in principle be reconstructed from A_{ij}; we will sketch how in module 7.

We close this module with an alternative way of encoding the Cartan subalgebra data for a given Lie algebra. The **Dynkin diagram** for a Lie algebra L with Cartan matrix $\hat{A}$ is constructed as follows.

- For each simple root, place an open dot $\circ$.
- Fill in (darken $\bullet$) dots corresponding to short roots.
- Connect the ith and j^{th} dots with
$$A_{ij}A_{ji} = 4\cos^2\theta \in \{0,1,2,3\}$$
straight lines. Thus orthogonal simple roots ($\langle\overline{\alpha}_i,\overline{\alpha}_j\rangle = 0$ corresponding to $[e_{\overline{\alpha}_i}, e_{\overline{\alpha}_j}] = 0$) are disconnected, while simple root pairs separated by 120°, 135°, or 150° are respectively connected by one, two, or three lines.

The Dynkin diagram for a simple Lie algebra is fully connected, but can be divided into two pieces by cutting one single, double, or triple "bond." The Dynkin diagram for a semi-simple Lie algebra consists of disjoint pieces, each of which corresponds to a simple Lie algebra.

The rank-2 examples are shown in Fig. 5.10.

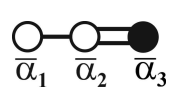

Figure 5.11: The Dynkin diagram for a rank-3 algebra [$B_3 = \mathrm{so}(5)$].

We can also reconstruct the Cartan matrix from a Dynkin diagram. A rank-3 example is shown in Fig. 5.11. $\overline{\alpha}_3$ is apparently a short root. $A_{12}A_{21} = 1$, so that $A_{12} = A_{21} = -1$. $\overline{\alpha}_1$ and $\overline{\alpha}_3$ are not connected, so that $A_{13} \propto A_{31} \propto \langle \overline{\alpha}_1, \overline{\alpha}_3 \rangle = 0$. Finally $A_{23}A_{32} = 2$, which implies that $\{A_{23}, A_{3,2}\} \in \{-1, -2\}$. Since $\overline{\alpha}_3$ is short, we conclude that

$$\frac{A_{23}}{A_{32}} = \frac{|\overline{\alpha}_2|^2}{|\overline{\alpha}_3|^2} = 2.$$

Therefore the Cartan matrix corresponding to Fig. 5.11 is

$$\hat{A}_{B_3} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix}. \tag{5.3.29}$$

This is the Cartan matrix for so(7) = B_3.

References

[1] Robert N. Cahn, *Semi-Simple Lie Algebras and Their Representations* (Benjamin/Cummings, Menlo Park, California, 1984).

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